Logical Limit Laws for Layered Permutations and Related Structures

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February 2022

Overview

Introduction: zero-one laws

Definition

A class C of structures in some first-order language admits a *zero-one law if, for any sentence* φ*, the probability that a randomly selected* C*-structure of size n satisfies* φ *converges asymptotically to zero or one as* $n \to \infty$.

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- Classical example: finite graphs [Glebskii et. al]

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- Classical example: finite graphs [Glebskii et. al]
- Convergence to zero or one is a rather strict requirement

Introduction: logical limit laws

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Introduction: logical limit laws

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- "Unlabeled limit law" — class of *unlabeled* structures admits a limit law
- "Labeled limit law" — class of *labeled* structures admits a limit law

Introduction: main results

Theorem

Convex linear orders and layered permutations admit both unlabeled and labeled limit laws. Compositions admit an unlabeled limit law.

Convex linear orders

Definition

Let $\mathcal{L}$ be the language containing two binary relations: $<$ and E. A convex linear order is an $\mathcal{L}$-structure satisfying:

- $<$ is a total order on points
- E is an equivalence relation
- $x\ E\ z, x < y < z \Rightarrow z\ E\ x, y$

Definition

Let $\mathfrak{C}$ be a convex linear order. Define $\widehat{\mathfrak{C}}$ to be the convex linear order obtained by adding one additional point to the last class of $\mathfrak{C}$.

Sum operators

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Definition

For convex linear orders $\mathfrak{C}, \mathfrak{D}$, define $\mathfrak{C} \oplus \mathfrak{D}$ as the convex linear order placing $\mathfrak{D}$ $<$-after $\mathfrak{C}$.

Constructing convex linear orders

Lemma

Every finite convex linear order containing n points can be uniquely constructed by applying $\widehat{(-)}$ and/or $- \oplus \bullet$ to $\bullet$ repeatedly.

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Proof

Proceed by induction.

- Base case: $n = 1$ trivial
- When $n = 2$, two possible cases: $\mathfrak{C} \simeq \bullet \oplus \bullet$ or $\mathfrak{C} \simeq \widehat{\bullet}$
- In general: last class of $\mathfrak{C}$ contains one or more points. Apply $- \oplus \bullet$ or $\widehat{(-)}$ appropriately. $\square$

Ehrenfeucht–Fraïssé games

- Ehrenfeucht–Fraïssé game on two structures: back-and-forth game between players Spoiler and Duplicator in which corresponding points are marked on each structure
- In game of length k between $\mathfrak{A}$ and $\mathfrak{B}$, Duplicator has a winning strategy iff $\mathfrak{A}$ and $\mathfrak{B}$ agree on all sentences of quantifier depth at most k.
- Write $\mathfrak{A} \equiv_k \mathfrak{B}$ in this case

Equivalences

Lemma

Let $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}', \mathfrak{N}'$ be convex linear orders such that $\mathfrak{M} \equiv_k \mathfrak{N}$ and $\mathfrak{M}' \equiv_k \mathfrak{N}'$. The following equivalences hold:

- $\mathfrak{M} \oplus \mathfrak{M}' \equiv_k \mathfrak{N} \oplus \mathfrak{N}'$
- $\widehat{\mathfrak{M}} \equiv_k \widehat{\mathfrak{N}}$

Lemma

For a convex linear order $\mathfrak{M}$ and $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that for all $s, t > \ell$,

$$\bigoplus_{s} \mathfrak{M} \equiv_k \bigoplus_{t} \mathfrak{M}$$

The limit law

- Labeled limit laws: count all possible structures over $[n] := \{1, \ldots, n\}$ as $n \to \infty$
- Unlabeled: count all structures *up to isomorphism*

The limit law

- Labeled limit laws: count all possible structures over $[n] := \{1, \ldots, n\}$ as $n \to \infty$
- Unlabeled: count all structures *up to isomorphism*
- Finite linearly ordered structures have no nontrivial automorphisms, hence, no distinction in this case

The limit law

General idea:

- For first-order sentence φ (with quantifier rank k), associate a Markov chain M_φ
- States of M_φ are $\equiv_k$-classes
- Probability that randomly selected structure of size n satisfies φ is probability that M_φ is in a state that satisfies φ after n transitions

For an $\equiv_k$-class C, define

$$C \oplus \bullet := [\mathfrak{M} \oplus \bullet]_{\equiv_k}$$

and

$$\widehat{C} := [\widehat{\mathfrak{M}}]_{\equiv_k}$$

The limit law

For φ an $\mathcal{L}$-sentence (with quantifier depth k), construct a Markov chain M_φ as follows:

- Starting state: $[\bullet]_{\equiv_k}$
- From any $\equiv_k$-class C, there are two possible transitions out: to $C \oplus \bullet$ or $\widehat{C}$
- Each transition probability is $1/2$

Definition

A Markov chain M is *fully aperiodic* if there do not exist disjoint sets of M-states $P_0, P_1, \ldots, P_{d-1}$ for some $d > 1$ such that for every state in P_i, M transitions to a state in P_{i+1} with probability 1 (with P_{d-1} transitioning to P_0).

Lemma

Let M be a finite, fully aperiodic Markov chain with initial state S, and let $Pr^{n-1}(S, Q)$ denote the probability that M is in state Q after $n - 1$ steps. For any Q, $\lim_{n\to\infty} Pr^{n-1}(S, Q)$ converges.

The limit law

Theorem

M_φ is fully aperiodic for any first-order sentence φ.

Proof

Suppose M_φ were not fully aperiodic.

- There would exist disjoint sets of M_φ-states ($\equiv_k$-classes) $P_0, P_1, \ldots, P_{d-1}$ for $d > 1$ where every state in P_i, M_φ transitions to a state in P_{i+1} with probability 1 (P_{d-1} transitioning to P_0).
- Thus, for any $Q \in P_0$, $Q \oplus i\bullet$ is in P_0 iff $d \mid i$.
- By equivalence lemmas, this is not possible $\square$

The limit law

Theorem

Convex linear orders admit a logical limit law.

Proof

Fix a first-order sentence φ, and consider M_φ.

- For each state S in M_φ, either each structure in S satisfies φ or no structures in S satisfy φ.
- Let S_φ denote the set of states in M_φ for which all structures in that state satisfy φ.
- $\widehat{(-)}$ and $- \oplus \bullet$ are well-defined on $\equiv_k$-classes, hence, moving $n-1$ steps in M_φ is equivalent to starting with any structure in the current state, applying $\widehat{(-)}$ or $- \oplus \bullet$ $n-1$ times, and taking the $\equiv_k$-class.

Proof (continued)

- The probability that after n steps, M_φ is in a state of S_φ equals probability that uniformly randomly selected structure of size n satisfies φ
- Suffices to show that $\lim_{n\to\infty} \sum_{Q\in S_\varphi} Pr^{n-1}(\bullet, Q)$ converges, which follows from Markov chain lemma $\square$

Transfer lemmas

Fix languages $\mathcal{L}_0, \mathcal{L}_1$ and classes C_0, C_1 of $\mathcal{L}_0, \mathcal{L}_1$ structures respectively.

Lemma

Let f be a map from the set of $\mathcal{L}_0$-structures to the set of $\mathcal{L}_1$-structures, and g a map from the set of $\mathcal{L}_0$-sentences to the set of $\mathcal{L}_1$-sentences such that, for any C_0-structure $\mathfrak{M}$ and $\mathcal{L}_0$-sentence φ:

1. $\mathfrak{M} \models \varphi \iff f(\mathfrak{M}) \models g(\varphi)$
2. f is a bijection between C_0 and C_1 structures of size n
3. The class C_1 admits a logical limit law

Then, C_0 admits a logical limit law as well.

Uniform interdefinability

Definition

Classes $\mathcal{C}_0$ and $\mathcal{C}_1$ of structures (over a common domain of $[n]$) are said to be *uniformly interdefinable* if there exists a map $f_I : \mathcal{C}_0 \to \mathcal{C}_1$ (bijective on structures), along with formulae $\varphi_{R_{0,i}}, \varphi_{R_{1,i}}$ for each relation $R_{0,i}$ in $\mathcal{L}_0$ and $R_{1,i}$ in $\mathcal{L}_1$ such that, for each $\mathfrak{M}_0$ in $\mathcal{C}_0$ and $\mathfrak{M}_1$ in $\mathcal{C}_1$:

- $\mathfrak{M}_0 \models R_{0,i}(\bar{x}) \iff f_I(\mathfrak{M}_0) \models \varphi_{R_{0,i}}(\bar{x})$
- $\mathfrak{M}_1 \models R_{1,i}(\bar{x}) \iff f_I^{-1}(\mathfrak{M}_1) \models \varphi_{R_{1,i}}(\bar{x})$

Uniform interdefinability

Theorem

Let C_0, C_1 be uniformly interdefinable classes of $\mathcal{L}_0$, $\mathcal{L}_1$ structures. If C_1 admits a logical limit law, C_0 admits one as well.

Proof

Take the transfer maps f, g to be:

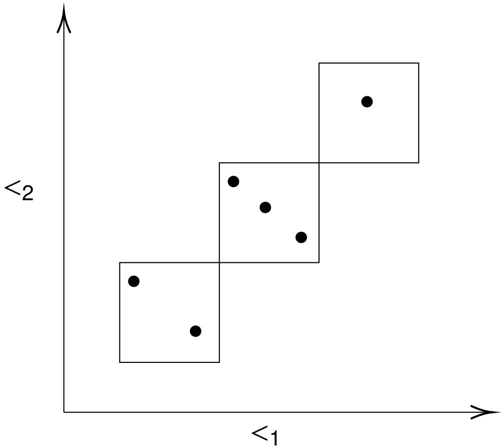
- $f = f_I$
- g is the map sends an $\mathcal{L}_0$-sentence to the $\mathcal{L}_1$-sentence with each ocurrence of $R_{0,i}$ replaced with $\varphi_{R_{0,i}}$

$\square$

Layered permutations

- Permutations can be viewed as structures in the language $\mathcal{L} = \{<_1, <_2\}$ with two linear orders. The order $<_1$ gives the unpermuted order of the points (before applying the permutation) and $<_2$ describes the points in permuted order.
- *Blocks* are maximal subsets which are monotone $<_1/<_2$-intervals
- A *layered permutation* is composed of increasing blocks, each of which contains a decreasing permutation

Layered permutations



Layered permutations

Lemma

Layered permutations and convex linear orders are uniformly interdefinable.

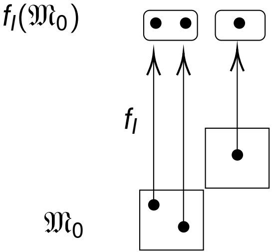
Proof

Define f_l to be the map taking blocks of a layered permutation to classes of a convex linear order, and points in an order-preserving manner. The relations $<_1$ and $<_2$ are rewritten as:

- $\varphi_{<_1} : a <_1 b \rightsquigarrow a < b$
- $\varphi_{<_2} : a <_2 b \rightsquigarrow (a\ E\ b \wedge b < a) \vee (\neg(a\ E\ b) \wedge a < b)$

$\square$

Layered permutations



Layered permutations

Theorem

Layered permutations admit a logical limit law.

Proof

Layered permutations are uniformly interdefinable with convex linear orders. Because convex linear orders admit a logical limit law, layered permutations admit one as well. □

Fractured orders

- Let $\mathcal{L}_0 = \{E, <\}$ be the language of convex linear orders
- Define $\mathcal{L}_1 = \{E, <_1, <_2\}$
- Fractured orders take a convex linear order $<$ and break it into two parts: $<_1$ *between* E-classes, and $<_2$ *within* E-classes.

Definition

A *fractured order* is an $\mathcal{L}_1$-structure satisfying:

1. $<_1$, $<_2$ are partial orders
2. E is an equivalence relation
3. Distinct points a, b are $<_1$-comparable iff they **are not** E-related
4. Distinct points a, b are $<_2$-comparable iff they **are** E-related
5. $a \, E \, a'$, $a <_1 b \Rightarrow a' <_1 b$ (convexity)

We denote the class of all finite fractured orders by $\mathcal{F}$.

Fractured orders

Theorem

Fractured orders and convex linear orders are uniformly interdefinable.

Proof

Define $f_I : \mathcal{F} \to \mathcal{C}_0$ such that:

- $\mathfrak{M}_1 \models a\ E\ b \iff f_I(\mathfrak{M}_1) \models a\ E\ b$
- $\mathfrak{M}_1 \models a <_1 b \iff f_I(\mathfrak{M}_1) \models \neg a\ E\ b \wedge a < b$
- $\mathfrak{M}_1 \models a <_2 b \iff f_I(\mathfrak{M}_1) \models a\ E\ b \wedge a < b$

This map satisfies the requirements for uniform interdefinability. $\square$

Reducts and limit laws

Lemma

Let $\mathcal{L}$ be a language and $\mathcal{L}' \subset \mathcal{L}$. Given a class C of $\mathcal{L}$-structures which admits a logical limit law, any class C' of $\mathcal{L}'$-structures which expand uniquely to C-structures also admits a logical limit law.

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Proof

Construct the transfer maps *f* and *g* from earlier:

- *f* is taken to be the map sending a structure in C' to its unique expansion in C
- This expansion is unique, hence *f* is bijective on structures of size *n* for all *n*
- *g* is given by the identity map on formulas

□

Compositions

- *Compositions* are structures in the reduct $\mathcal{L}_2 \subset \mathcal{L}_1$ given by $\mathcal{L}_2 = \{E, \prec_1\}$
- Order defined on equivalence classes, but not on points within each class

Compositions

Lemma

Every composition expands uniquely to a fractured order, up to isomorphism.

Proof

There is a unique way to linearly order each E-class individually. Because ordering these classes determines $<_2$, there is a unique way to define $<_2$ on any composition, expanding it to a fractured order. $\square$

Compositions

Theorem

The class of compositions admit an unlabeled logical limit law.

Proof

The language of compositions is a reduct of the language of fractured orders, and every composition expands uniquely to a fractured order. The class of fractured orders admits a logical limit law, therefore, by the previous lemma, compositions admit a limit law as well. $\square$

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